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NOTE

Rates of Best Rational Approximation of Analytic Functions

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Let E be a compact set in the extended complex plane $\overline{\mathbb{C}}$ and let f be holomorphic on E. Denote by ρ_n the distance from f to the class of all rational functions of order at most n, measured with respect to the uniform norm on E. We obtain results characterizing the relationship between estimates of $\liminf_{n\to\infty} \rho_n^{1/n}$ and $\limsup_{n\to\infty} \rho_n^{1/n}$. © 2000 Academic Press

Let f be holomorphic on a compact set E in the extended complex plane $\overline{\mathbb{C}}$ and let ρ_n be the error in best approximation to f in the supremum norm on E by rational functions of order at most n. By the well-known theorem of Walsh [6], if f is holomorphic on $\overline{\mathbb{C}}\setminus F$, where F is a compact set in $\overline{\mathbb{C}}$ such that $F \cap E = \varnothing$, then

$$\limsup_{n\to\infty} \rho_n^{1/n} \leqslant 1/\rho, \tag{1}$$

where $\rho = \exp(1/C(E, F))$ and $C(E, F)$ denotes the condenser capacity associated with (E, F) (see, for example, [5]). We mention the paper of Parfenov [1] (the case when E is the unit disk) and the paper of the author [2] (the general case), where methods in the theory of Hankel operators are used to characterize the rate of convergence of the product $\rho_1 \rho_2 \cdots \rho_n$ to zero:

$$\limsup_{n\to\infty} (\rho_1 \rho_2 \cdots \rho_n)^{1/n^2} \leqslant 1/\rho \tag{2}$$

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(see also [4]). Walsh's inequality (1) and the following upper estimate for $\liminf_{n\to\infty} \rho_n^{1/n}$

$$\liminf_{n\to\infty} \rho_n^{1/n} \leqslant 1/\rho^2$$

are immediate consequences of the inequality (2). It is also proved in [2], that if

$$\limsup_{n\to\infty} \rho_n^{1/n} = \frac{1}{\rho}, \tag{3}$$

then

$$\liminf_{n\to\infty} \rho_n^{1/n} = 0. \tag{4}$$

The present note is devoted to results generalizing (3) and (4) and describing the relationship between estimates of $\limsup_{n\to\infty} \rho_n^{1/n}$ and $\liminf_{n\to\infty} \rho_n^{1/n}$.

THEOREM 1. (i) *If*

$$\limsup_{n\to\infty} \rho_n^{1/n} \geqslant \frac{\lambda}{\rho}, \qquad \frac{1}{\rho} \leqslant \lambda \leqslant 1,$$

then

$$\liminf_{n\to\infty} (\rho_1 \rho_2 \cdots \rho_n)^{1/n^2} \leqslant \frac{1}{\rho} \left(\frac{1}{\rho}\right)^{1/4(\sqrt{\log \lambda/\log(1/\rho)} - \sqrt{\log(1/\rho)/\log \lambda})^2} \tag{5}$$

and

$$\liminf_{n\to\infty} \rho_n^{1/n} \leqslant \frac{1}{\rho^2} \left(\frac{1}{\rho}\right)^{1/2(\sqrt{\log \lambda/\log(1/\rho)} - \sqrt{\log(1/\rho)/\log \lambda})^2}. \tag{6}$$

(ii) *If*

$$\liminf_{n\to\infty} \rho_n^{1/n} \geqslant \frac{\lambda}{\rho}, \qquad 0 < \lambda \leqslant 1/\rho,$$

then

$$\limsup_{n\to\infty} \rho_n^{1/n} \leqslant \frac{\lambda}{\rho} \left(\frac{1}{\rho}\right)^{-\sqrt{(\log \lambda/\log(1/\rho))^2 - 1}}.$$

In particular, if

$$\liminf_{n\to\infty} \rho_n^{1/n} \geqslant \frac{1}{\rho^2},$$

then

$$\lim_{n\to\infty} \rho_n^{1/n} = \frac{1}{\rho^2}.$$

Proof. We prove (i). The second part (ii) of Theorem 1 follows directly from (i). Denote by Λ a sequence of positive integers such that

$$\lim_{n\to\infty,\, n\in\Lambda} \rho_n^{1/n} \geqslant \frac{\lambda}{\rho}, \tag{7}$$

where $1/\rho \leqslant \lambda \leqslant 1$. Fix an arbitrary $0 \leqslant \theta \leqslant 1$. Choose a sequence of integers $\{k_n\}$, $n = 1, 2, ...,$ such that $1 \leqslant k_n \leqslant n$,

$$\lim_{n\to\infty} k_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{k_n}{n} = \theta.$$

Since the sequence $\{\rho_n\}$, $n = 1, 2, ...,$ is nonincreasing,

$$(\rho_1 \cdots \rho_{k_n})\, \rho_n^{n-k_n} \leqslant \rho_1 \rho_2 \cdots \rho_n.$$

From this and from the relations (2) and (7), we obtain

$$\limsup_{n\to\infty,\, n\in\Lambda} (\rho_1 \rho_2 \cdots \rho_{k_n})^{1/k_n^2} \leqslant \left(\frac{1}{\rho}\right)^{1/\theta} \lambda^{-1/\theta^2 + 1/\theta},$$

which implies that

$$\liminf_{n\to\infty} (\rho_1 \rho_2 \cdots \rho_n)^{1/n^2} \leqslant \left(\frac{1}{\rho}\right)^{1/\theta} \lambda^{-1/\theta^2 + 1/\theta}.$$

Substituting $\theta = 2(\log(1/\rho)/\log \lambda + 1)^{-1}$, we get (5). It remains to remark that (6) follows immediately from (5). ∎

We now point out results characterizing the rate of decrease of the best approximation errors ρ_n of entire functions. The following estimates are established in [2] (see also [3]):

If f is an entire function of finite order $\sigma \geqslant 0$, then

$$\limsup_{n\to\infty} \frac{\log(\rho_1\rho_2\cdots\rho_n)}{n^2\log n} \leqslant -\frac{1}{\sigma},$$

$$\limsup_{n\to\infty} \frac{\log \rho_n}{n\log n} \leqslant -\frac{1}{\sigma},$$

and

$$\liminf_{n\to\infty} \frac{\log \rho_n}{n\log n} \leqslant -\frac{2}{\sigma}.$$

As above, it is easy to prove the following assertion.

THEOREM 2. (i) *If*

$$\limsup_{n\to\infty} \frac{\log \rho_n}{n\log n} \geqslant -\frac{\lambda}{\sigma}, \qquad 1 \leqslant \lambda \leqslant 2,$$

then

$$\liminf_{n\to\infty} \frac{\log(\rho_1\rho_2\cdots\rho_n)}{n^2\log n} \leqslant -\frac{1}{\sigma} - \frac{(\lambda-2)^2}{4\sigma(\lambda-1)}$$

and

$$\liminf_{n\to\infty} \frac{\log \rho_n}{n\log n} \leqslant -\frac{2}{\sigma} - \frac{(\lambda-2)^2}{2\sigma(\lambda-1)}.$$

(ii) *If*

$$\liminf_{n\to\infty} \frac{\log \rho_n}{n\log n} \geqslant -\frac{\lambda}{\sigma}, \qquad 2 \leqslant \lambda \leqslant \infty,$$

then

$$\limsup_{n\to\infty} \frac{\log \rho_n}{n\log n} \leqslant -\frac{\lambda - \sqrt{\lambda^2 - 2\lambda}}{\sigma}.$$

In particular, if

$$\liminf_{n\to\infty} \frac{\log \rho_n}{n\log n} \geqslant -\frac{2}{\sigma}$$

then

$$\lim_{n \to \infty} \frac{\log \rho_n}{n \log n} = -\frac{2}{\sigma}.$$

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